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Controlling the self-organizing dynamics in a sandpile model on complex networks by failure tolerance

Junjian Qi[1] and Stefan Pfenninger[2]

[1] *Energy Systems Division, Argonne National Laboratory - 9700 South Cass Avenue, Lemont, IL, USA*
[2] *Department of Civil and Environmental Engineering and Grantham Institute, Imperial College London London, UK*

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Abstract – In this paper, we propose a strategy to control the self-organizing dynamics of the Bak-Tang-Wiesenfeld (BTW) sandpile model on complex networks by allowing some degree of failure tolerance for the nodes and introducing additional active dissipation while taking the risk of possible node damage. We show that the probability for large cascades significantly increases or decreases respectively when the risk for node damage outweighs the active dissipation and when the active dissipation outweighs the risk for node damage. By considering the potential additional risk from node damage, a non-trivial optimal active dissipation control strategy which minimizes the total cost in the system can be obtained. Under some conditions the introduced control strategy can decrease the total cost in the system compared to the uncontrolled model. Moreover, when the probability of damaging a node experiencing failure tolerance is greater than the critical value, then no matter how successful the active dissipation control is, the total cost of the system will have to increase. This critical damage probability can be used as an indicator of the robustness of a network or system.

Introduction. – Self-organizing criticality (SOC) [1–7] has been seen in many natural and engineering systems, such as earthquakes [3], forest fires [4], and power grids [8–10], which means that a system can self-organize toward the critical point with power-law–distributed event sizes. SOC has been applied to study and understand the cascading failures in complex systems [11,12], which are complicated sequences of dependent outages and can take place in electric power systems [8–10,13,14], the Internet [15], the road system [16], and in social and economic systems [17]. Because self-organizing systems can respond via feedback mechanisms to the control strategies applied to them, successfully controlling them is very difficult.

The sandpile model is a prototypical theoretical model exhibiting avalanche behavior that can be used to study cascading failures and self-organizing dynamics on complex networks. It consists of a network of nodes each holding a number of sand grains. Each node has a maximum amount of sand it can hold (its capacity). If more sand is added to a node already at capacity, it topples and distributes sand to its neighboring nodes (which may in turn topple if they exceed their capacity). In [18–21], control strategies are proposed by adjusting where a cascade tends to begin (*i.e.* where the sand that launches the cascade lands). But in real systems, the cascade's origin is usually uncontrollable. Thus, rather than trying to control where a cascade begins, control strategies need to be able to deal with a cascade once it has begun. For example in [22], the self-organizing dynamics of a sandpile model are controlled by introducing immunization to some vertices, which can thus absorb an infinite amount of sand.

In this paper, we propose a more realistic strategy for controlling the self-organizing dynamics of a system by making use of the failure tolerance of components in the system. Generally the components are designed to be able to sustain abnormal operating conditions for some time. For example, in power systems, which can be considered as a dual network that maps the transmission lines into nodes, the transmission lines can operate at an overloading power flow for a short time because they are designed to have some margin over the normal power flow operating conditions. Although there is a potential risk of damage, it is possible to help prevent the propagation of the cascading failures by allowing some components to temporarily work at abnormal operating conditions, thus securing time for the operators to perform proper control and recover the system back to its normal state.

BTW sandpile model. – This section describes the sandpile model and how we create cascades on it. We consider the Bak-Tang-Wiesenfeld (BTW) sandpile model [1–3] to represent the basic self-organizing dynamics of a system. The system consists of a network of n nodes that hold grains of sand. The topology of the network is fixed but the number of sand on each node changes in time. Let $k(i)$ and $\mathcal{N}(i)$ respectively be the degree and the set of neighbors of node i. Each node i holds a certain number z_i grains of sand and a node is called s-sand if it holds s grains of sand. The capacity of a node $K(i)$ is the maximum amount of sand it can hold and is set to $K(i) = k(i) - 1$ for node i. A node over capacity topples by shedding one sand to each of its neighbors.

On this network, cascades occur as follows: Drop a grain of sand on a node i, which is chosen uniformly at random and is called the root of the cascade, so that $z_i \to z_i + 1$. If this addition of sand does not bring the root over capacity (the root is initially an s-sand where s is less than the capacity), then the cascade is finished. Otherwise, the root topples by shedding one grain to each of its neighbors as $z_i \to z_i - k(i), z_j \to z_j + 1, j \in \mathcal{N}(i)$. Any node that now exceeds its capacity topples in the same way until all nodes are under or at capacity. Whenever a grain of sand moves from one node to another, it can potentially dissipate (disappear) with probability ϵ.

The size of a cascade is the number of toppling events, while the area of a cascade is the number of nodes that topple. We begin a new cascade by dropping a grain on a uniformly random root node. After a transient the system reaches a steady state in which the input and output of energy is balanced.

For simplicity we first consider the BTW process on a random 4-regular graph (*i.e.*, a random network of degree-four nodes). Furthermore, each node has a random initial load not exceeding its capacity.

Control by failure tolerance. – This section discusses how the sandpile model described above is modified to suppress cascades by failure tolerance. Similar to the basic sandpile model, a grain of sand is dropped on a node i (the root node) which is chosen uniformly at random so that its grains of sand $z_i \to z_i + 1$. If this addition of sand does not bring the root over capacity, then the cascade ends (again, this behavior is as in the basic model). Failure tolerance comes into play if the root node exceeds its capacity: it will not immediately topple, rather, it has the ability to operate for a short time above its capacity. This creates a time window during which some control strategy can be used to eliminate the overcapacity and return the system to its normal condition before the node topples and potentially causes a cascading failure. For example, in power systems this control strategy may be to shed some load.

Two additional steps are now taken which do not occur in the basic model.

First, because of its non-immediate toppling, the node operating above its capacity can sustain damage with a probability ϵ_{dam}. If a node is damaged, its grains of sand will be redistributed to its neighbors, the edges connecting it to its neighbors will be removed, and it will not be able to hold any sand (*i.e.*, it is considered non-functional and removed from the network). Furthermore, the degree and the capacity of its neighbors will decrease by 1 as $k(j) \to k(j) - 1, K(j) \to K(j) - 1, j \in \mathcal{N}(i)$, which in turn decreases the system's total capacity of holding sand. If all the edges of a node i are removed, all the sand it holds will be removed as $z_i \to 0$ if $k(i) = 0$.

Second, if the node over capacity does not get damaged, each grain of sand above capacity will dissipate with a probability ϵ_{act}. This can be considered an active shedding in response to the overcapacity, leading to decreased stress on the system and reducing the likelihood of a cascading failure. If this active dissipation successfully removes all extra $z_i - K(i)$ grains of sand, the node will now be at its capacity and thus the cascade stops; otherwise, it topples in a similar way to the basic BTW sandpile model without control, shedding its sand load to its neighbors, as described above.

If any additional node is now over capacity (because it received additional load due to a damaged or toppled neighbor), the same process applies again until all nodes are at or under capacity. Note that with the introduction of this control strategy, the topology of the network can change during a cascade due to node damage, which is very different from the basic BTW sandpile model without control described in the previous section. Consequently, it is possible that $z_i > K(i) + 1 = k(i)$. This occurs when the neighbor of a node i damages and the sand on this neighbor is redistributed to node i. In this case the grains of sand on node i will be $z_i \to 0$. For its neighbors, the grains of sand on node i will first be evenly distributed as $z_j \to z_j + \lfloor z_i/k(i) \rfloor, j \in \mathcal{N}(i)$ and the remaining $z_i \bmod k(i)$ grains of sand will be redistributed to the same number of randomly chosen neighboring nodes.

In order not to significantly change the dynamics of the system, the control is considered a temporary measure. When the cascade ends the network topology is recovered and the sand loads are reset to what they would be in the case without control. The control strategy considered here introduces a benefit through active dissipation, but also an additional risk from the possible damaging of overloaded nodes, with the consequent degradation of the network's overall capacity. For the results reported here we use a random 4-regular network of size $n = 10^5$ with $N = 10^7$ iterations of adding one grain of sand. Before launching N iterations of the active control case, N iterations of the uncontrolled case are performed to ensure that the system has passed through its transient state (the

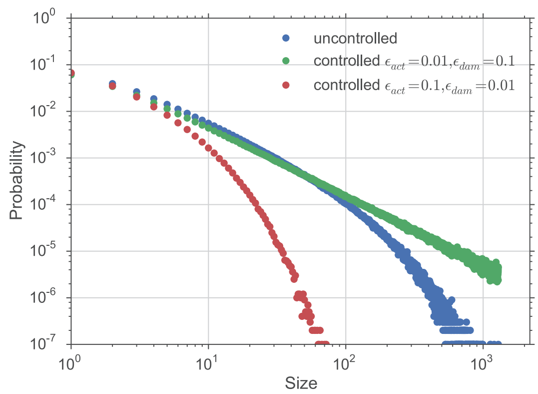


Fig. 1: (Colour on-line) Probability distribution of cascade size. $n = 10^5$, $N = 10^7$ and $\epsilon = 0.05$.

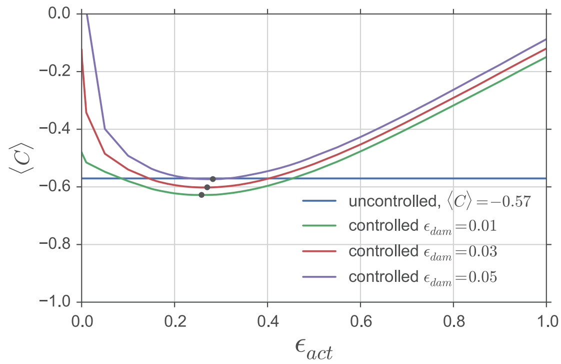


Fig. 2: (Colour on-line) Control effect under different parameters. The black dots represent optima (ϵ^*_{act}).

results from these initial N uncontrolled iterations are discarded in the further analysis). The resulting probability distributions of cascade sizes for different ϵ_{act} and ϵ_{dam} are shown in fig. 1. When the risk for node damage outweighs the active dissipation, the probability for large cascades significantly increases. When the active dissipation outweighs the risk for node damage, the probability for large cascades significantly decreases.

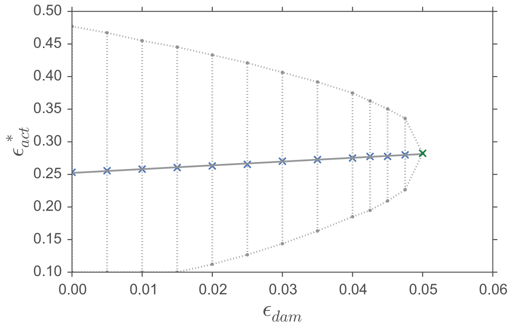
Cost and optimal control. – The total cost C_t for one cascade consists of three parts: the cost of the size of cascades C_{cas}, the cost of active sand dissipation C_{act}, and the cost of node damage C_{dam}. Therefore, $C_t = C_{cas} + C_{act} + C_{dam}$. Similar to [21], when cascade size is greater than zero $C_{cas} = c\,[\text{size}]^\alpha$ where $\alpha > 1$ and when cascade size is zero there is a benefit of 1 (*i.e.* $C_{cas}[\text{size} = 0] = -1$). As in [21], this benefit defines the scale of costs and in real systems can be profit on uneventful days for infrastructures and investment portfolios. Here c and α are chosen as 0.005 and 1.5. For C_{act} and C_{dam}, we set the cost for each sand dissipation and each node damage respectively as 0.6 and 1.0, which correspond to the cost of cascade size of 24 and 34. In other words, equivalently $C_{act} = n_{act}\, c\, [24]^\alpha$ and $C_{dam} = n_{dam}\, c\, [34]^\alpha$, where n_{act} and n_{dam} are respectively the grains of actively dissipated sand and the number of damaged nodes in one cascade.

By setting the cost in this way, the cost for one node damage is greater than that for each sand dissipation and the cost for each sand dissipation is greater than that for cascade size one. This is reasonable in the model because the damage of a node can greatly degrade the capacity of the network for not being able to hold sand any more and for causing its neighbors' capacity to decrease, the control will cause intentional extra sand dissipation, while cascade mainly transfers grains of sand from one node to another and the only possible loss is the very weak sand dissipation with probability ϵ.

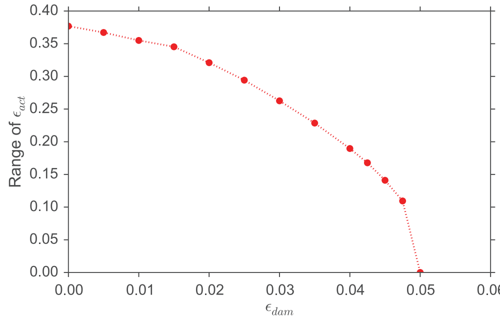
This is also realistic in real systems since the damage of a component can cause great economic loss due to loss of functionality of that component and the cost of repair after a cascade ends. The control strategy that actively dissipates load will intentionally make some load disappear. This dissipated load corresponds to an economic loss, while a cascade itself does not directly cause economic loss. For example, in power systems, if a transmission line is damaged during a cascading blackout it usually cannot be recovered until the cascade ends. It therefore cannot transmit power at the time when it is most needed to do so. Furthermore, it then has to undergo costly repairs or even replacement. By contrast, the control can be implemented by actively shedding some load, preventing a further overloading of transmission lines. Shedding load will cause the disruption of some consumers and will surely cause economic loss, but it does not cause component damage or influence the functioning of the transmission network. Furthermore, during a cascade caused by the tripping of one power line, the power generated by the generators can still be transmitted to the consumers through other transmission lines, thus not directly causing economic loss.

Given a fixed ϵ_{dam} we can find an optimal control parameter ϵ^*_{act} that minimizes the average total cost $\langle C_t \rangle$. Figure 2 shows the cost under different ϵ_{dam} and ϵ_{act}, obtained by sampling over a discrete set of values for ϵ_{dam} and ϵ_{act}. For each curve ϵ_{dam} is fixed and there is non-trivial optimal ϵ_{act} for which the total cost is minimized (to determine them, a third-degree polynomial is fitted to the sampled cost curves to obtain smooth curves). The cost for the basic model without control is also plotted for comparison. When ϵ_{act} is increased, the control becomes more successful and it is more easily possible to stop the propagation of cascades, thus decreasing the cost of cascades. However, the control itself has a cost and thus the increased ϵ_{act} will increase the cost of control. Therefore, there is an optimal ϵ_{act} which will guarantee minimum total cost. From fig. 2 it is also seen that an increased risk for performing the control, *i.e.* higher values for ϵ_{dam} will require higher optimal ϵ_{act}.

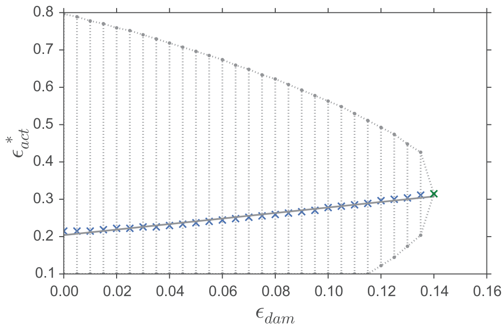
For any fixed ϵ_{dam}, the ϵ_{act} for optimal control is shown in fig. 3(a). We can see that the optimal ϵ_{act} will slightly increase when ϵ_{dam} increases. This is because



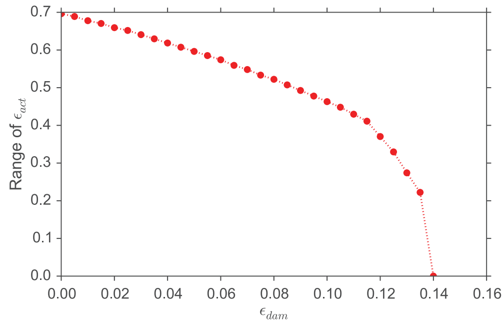
(a) Optimal control (4-regular network).



(b) Range width of ϵ_{act} that reduces the total cost (4-regular network).



(c) Optimal control (scale-free network).



(d) Range width of ϵ_{act} that reduces the total cost (scale-free network).

Fig. 3: (Colour on-line) Optimal control and range width of ϵ_{act} that reduces the total cost for both the 4-regular and scale-free case. (a) and (c): the points marked in blue are the optimal control values (ϵ^*_{act}), and the vertical range indicates those values of ϵ_{act} for which the controlled case has lower cost than the uncontrolled case. The point marked in green is the critical ϵ_{dam} where the range becomes zero.

the increased risk of damage requires more successful control in order to limit the total cost. Furthermore, with the increase of ϵ_{dam} the range of ϵ_{act}, denoted by $[\epsilon^{min}_{act}(\epsilon_{dam}), \epsilon^{max}_{act}(\epsilon_{dam})]$, in which the total cost decreases will shrink and finally disappear, indicating that no matter how one adjusts the ϵ_{act} (how successful the active dissipation control is) the total cost will surely increase after the control is added. Here $\epsilon^{min}_{act}(\epsilon_{dam})$ and $\epsilon^{max}_{act}(\epsilon_{dam})$ respectively denote the minimum and maximum ϵ_{act} under ϵ_{dam} that can guarantee a decreased total cost compared with the uncontrolled case. The critical ϵ_{dam} that corresponds to zero range width ($\epsilon^{max}_{act}(\epsilon_{dam}) = \epsilon^{min}_{act}(\epsilon_{dam})$) for ϵ_{act} can be used as an indicator of the robustness of the network. It is denoted by ϵ^*_{dam} and the bigger it is the more robust the network is.

The range width $W = \epsilon^{max}_{act}(\epsilon_{dam}) - \epsilon^{min}_{act}(\epsilon_{dam})$ is shown in fig. 3(b). It is seen that W first decreases approximately linearly and then drops rapidly after the ϵ_{dam} exceeds some value (0.04 in our case). The critical ϵ^*_{dam} corresponding to zero W is around 0.05, which is a very small value and indicates that the 4-regular network we are considering is not very robust. When ϵ_{dam} is greater than 0.05, no matter how successful the active dissipation control is, the total cost of the system will have to increase.

Influence of network structures. – In this section, we examine how the random network described earlier compares to a scale-free network [3]. A scale-free network has some high-degree nodes and for the same ϵ_{act}, high-degree nodes are easier to control, but the effect of damaging a high-degree node can also be significant. Therefore, it is not obvious whether it is easier or harder to control a scale-free network.

For this comparison, a scale-free network is generated with the Barabási-Albert preferential attachment model [23] as implemented in NetworkX [24], with $m = 2$ (where m is the number of edges to attach from a new node to existing nodes), resulting in a mean degree of 4 to match the 4-regular graph. Again, the initial load on each node is randomly assigned (lesser or equal to its capacity). As in the 4-regular case, $n = 10^5$ and $N = 10^7$. The control proceeds is exactly the same manner as described above for the 4-regular case.

The resulting optimal control and range width of ϵ_{act} that reduce the total cost are shown in figs. 3(c) and (d).

It is seen that for the same ϵ_{dam} the range of ϵ_{act} reducing the total cost for the scale-free network is larger than that for the random network. The critical damage probability for the scale-free network is around 0.14, which is also much higher than the random network. From these results, it appears that the scale-free network is more robust to random failures, which is consistent with the conclusion in [25].

Similar to [25], the robustness of scale-free networks to random failures should also root in their extremely inhomogeneous connectivity distribution. Power-law distribution implies that the majority of nodes have only a few edges, nodes with small connectivity will be influenced with much higher probability. Besides, being easier to control for high-degree nodes seems to play a more important role than the more significant effect of their damage.

Conclusion. – In this paper we propose a strategy to control the self-organizing dynamics of the BTW sandpile model on complex networks by allowing the failure tolerance of some nodes. For the control we consider here there is benefit from active dissipation and also additional risk from the damage of over-capacity nodes and the consequent degradation of the network's overall capacity. By considering the additional risk from node damage, a non-trivial optimal active dissipation control strategy which minimizes the total cost can be obtained. We show that due to the potential additional risk from node damage the introduced control can only decrease the total cost under some conditions, when compared with the uncontrolled model. Also, when the probability for the damage of a node experiencing failure tolerance is greater than the critical value, no matter how successful the active dissipation control is, the total cost of the system will have to increase. This critical damage probability can be used as an indicator of the robustness of a network or system and can be further used to compare the differing structures of networks in different fields. Possible applications of the model discussed in this paper, for which further work is needed, include physical [8–14], economic [17], and ecological networks [26,27].

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